LAST TIME: Line Integrals

- Fundamental Theorem of Line Integrals: Given curve C parameterized by $\vec{r}(t)$ on $[a,b]$ and f a function with continuous partial derivatives. Then, $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ where C is oriented from $\vec{r}(a)$ to $\vec{r}(b)$.
- Recall: Switching the orientation of curve C negates the corresponding line integral, i.e. $\int_{-C} \vec{v}\cdot d\vec{r} = -\int_C \vec{v}\cdot d\vec{r}$.

Ex #1: Compute $\int_C \vec{v}\cdot d\vec{r}$ for $\vec{v} = \langle \sin(y), x\cos(y) + \cos(z), -y\sin(z)\rangle$ and curve C parameterized by $\vec{r}(t) = \langle \sin(t), t, 2t\rangle$ on $[0, \frac{\pi}{2}]$.

First, we check $\vec{v}$ is conservative: $\leftarrow$ check that FTLI applies

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial y}[\sin(y)] = \cos(y) \qquad \frac{\partial}{\partial z}[v_x] = \frac{\partial}{\partial z}[\sin(y)] = 0 \;\overset{\because}{\text{———}}\; \frac{\partial}{\partial x}[v_z] = \frac{\partial}{\partial x}[-y\sin(z)] = 0$$

$$\frac{\partial}{\partial x}[v_y] = \frac{\partial}{\partial x}[x\cos(y) + \cos(z)] = \cos(y) \qquad \frac{\partial}{\partial z}[v_y] = \frac{\partial}{\partial z}[x\cos(y) + \cos(z)] = -\sin(z)$$

$$\frac{\partial}{\partial y}[v_z] = \frac{\partial}{\partial y}[-y\sin(z)] = -\sin(z)$$

$\therefore$ by a previous result, $\vec{v}$ is conservative, i.e. $\vec{v} = \nabla f$ for some function f.

Next, we compute such a potential function:

$$\frac{\partial f}{\partial x} = \sin(y), \quad \frac{\partial f}{\partial y} = x\cos(y) + \cos(z), \quad \frac{\partial f}{\partial z} = -y\sin(z)$$

$$f(x,y,z) = \int \frac{\partial f}{\partial z}\, dz = \int -y\sin(z)\, dz = y\cos(z) + C(x,y)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[y\cos(z) + C(x,y)] = \frac{\partial C}{\partial x} = \sin(y) \qquad \therefore C(x,y) = \int \frac{\partial C}{\partial x}\, dx = \int \sin(y)\, dx = x\sin(y) + D(y)$$

Hence, $f(x,y,z) = y\cos(z) + C(x,y) = y\cos(z) + x\sin(y) + D(y)$

$\therefore x\cos(y) + \cos(z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y\cos(z) + x\sin(y) + D(y)] = \cos(z) + x\cos(y) + D'(y)$

$\therefore D'(y) = 0$ so $D(y) = E$ is constant.

$\therefore f(x,y,z) = y\cos(z) + x\sin(y)$ is a potential for $\vec{v}$, setting $E = 0$.

$\therefore$ We may express $\int_C \vec{v}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$.

Now, $\vec{r}(b) = \vec{r}(\frac{\pi}{2}) = \langle \sin(\frac{\pi}{2}), \frac{\pi}{2}, 2\cdot\frac{\pi}{2}\rangle = \langle 1, \frac{\pi}{2}, \pi\rangle$ and $\vec{r}(a) = \vec{r}(0) = \langle \sin(0), 0, 2\cdot 0\rangle = \langle 0,0,0\rangle$.

Hence, $\int_C \vec{v}\cdot d\vec{r} = f(1, \frac{\pi}{2}, \pi) - f(0,0,0) = (\frac{\pi}{2}\cos(\pi) + 1\cdot\sin(\frac{\pi}{2})) - (0\cdot\cos(0) + 0\cdot\sin(0)) = \frac{\pi}{2}(-1) + 1 - 0 = 1 - \frac{\pi}{2}$ $\blacksquare$

Independence of Paths for Line Integrals of Conservative Vector Fields

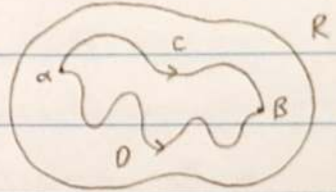
- PROP: Suppose C and D are two paths between the same endpoints α and β, and suppose $\vec{v}$ is conservative. Then,

$\int_C \vec{v} \cdot d\vec{r} = \int_D \vec{v} \cdot d\vec{r}$.

Proof Apply FTLI: $\int_C \vec{v} \cdot d\vec{r} = f(\beta) - f(\alpha) = \int_D \vec{v} \cdot d\vec{r}$ where $\vec{v} = \nabla f$

- PROP: If $\vec{v}$ satisfies $\int_C \vec{v} \cdot d\vec{r} = \int_D \vec{v} \cdot d\vec{r}$ for all C, D paths between the same endpoints on some open region R and if the components of $\vec{v}$ are all continuous on R, then $\vec{v}$ is conservative.

Proof Fix any point α in R.



Define $f(\beta) = \int_\alpha^\beta \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot d\vec{r}$ where C is any curve from α to β.

By independence of paths, f is well defined. Moreover, $\nabla f = \vec{v}$. ▨

↳ exercise, use the FTC

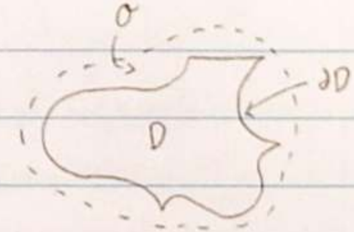
- Observation: If $\vec{v}$ is conservative and C is a closed curve (i.e. C starts and ends at the same point), then $\int_C \vec{v} \cdot d\vec{r} = 0$. Conversely, if $\int_C \vec{v} \cdot d\vec{r} = 0$ for all closed C, then $\vec{v}$ is conservative.

↳ exercise (hint: independence of paths)

– SECTION 16.4: Green's Theorem –

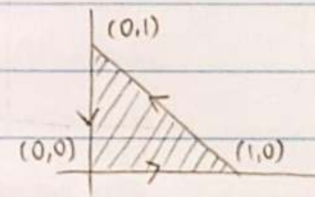
- IDEA: In some special cases, line integrals can be computed via double integrals.

- PROP (Green's Theorem): Let D be a region in $\mathbb{R}^2$ with a piecewise-smooth boundary curve ∂D. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on some open region $\mathcal{O}$ containing D, then we have
$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$



* For this theorem to hold, ∂D needs the positive orientation.

Ex#1: Compute $\int_C x^4\,dx + xy\,dy$ for C the curve positively oriented around the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.



* This would be monsterous normally, because the curve is split into 3 pieces.

By Green's Theorem, $\int_{\partial D} x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[x^4]\right) dA = \iint_D y\,dA.$

Note that $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$, so:

$\iint_{\partial D} x^4\,dx + xy\,dy = \iint_D y\,dA = \int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx = \int_{x=0}^{1} \frac{1}{2}\left[y^2\right]_{y=0}^{1-x} dx = \frac{1}{2}\int_{x=0}^{1} ((1-x)^2 - 0)\,dx$ $(u = 1-x\ \ du = -dx)$

$= -\frac{1}{2}\cdot\frac{1}{3}\left[(1-x)^3\right]_{x=0}^{1} = -\frac{1}{6}\left((1-1)^3 - (1-0)^3\right) = -\frac{1}{6}(-1) = \frac{1}{6}$ ▨

* Reminder: Green's Theorem only works when the curve is a simple, closed curve in the plane $\mathbb{R}^2$.

Ex#2: Compute $\int_C (\overbrace{3y - e^{\sin(x)}}^{\text{nasty}})\,dx + (\overbrace{7x + \sqrt{y^4+1}}^{\text{nasty}})\,dy$ for C the circle $x^2+y^2=9$.

$\int_{\partial D} (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy \overset{\text{Green's Theorem}}{=} \iint_D \left(\frac{\partial}{\partial x}[7x + \sqrt{y^4+1}] - \frac{\partial}{\partial y}[3y - e^{\sin(x)}]\right) dA$

$= \iint_D (7-3)\,dA = 4\iint_D dA = 4\,\text{Area}(D) = 4\pi(3)^2 = 36\pi$ ▨